The Measure of Arbitrage

Equivalent Martingale Measures, Girsanov's Theorem, and the Probabilistic Foundations of Black–Scholes Pricing

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Abstract

In the previous paper, "The Frontier of Certainty," the Black–Scholes option pricing equation was derived strictly through continuous dynamic replication [10, 2, 9], demonstrating that local delta-hedging eliminates equity diffusion risk and causes the physical stock drift μ to drop out of the valuation differential equation entirely. This note addresses the dual, probabilistic resolution of that same price. If dynamic replication yields a no-arbitrage price independent of μ, under what probability measure is that price evaluated when expressed as a discounted expected payoff, and why does μ disappear? We show that calculating derivative prices as discounted expectations requires evaluating payoffs under an equivalent martingale measure $\mathbb{Q}$ rather than the physical measure $\mathbb{P}$ [4, 5]. Using Girsanov's Theorem [3], we construct the Radon–Nikodym density process Z_t that absorbs the market price of risk $\theta = (\mu - r)/\sigma$, shifting the stock's expected rate of return from μ to the risk-free rate r while preserving volatility, path continuity, and null sets [8, 11]. Under $\mathbb{Q}$, discounted traded asset prices become true martingales, enabling the pricing of derivative claims via discounted conditional expectations [1]. Evaluating this expectation directly for a European call option recovers the Black–Scholes formula and its exact d_1, d_2 parameters. Finally, using market completeness and the Martingale Representation Theorem, we show that risk-neutral expectation and dynamic replication are not two competing theories, but the same no-arbitrage price viewed from two mathematical perspectives.

1 Introduction and the Central Question

In the previous paper, "The Frontier of Certainty," derivative pricing was established on a purely deterministic foundation through continuous-time dynamic replication [10]. By constructing a self-financing portfolio consisting of the derivative, the underlying stock, and a risk-free money market account, instantaneous diffusion risk was completely eliminated [2, 9]. Absence of arbitrage demanded that this riskless hedged portfolio earn the risk-free rate of return r. This dynamic balancing yielded the Black–Scholes partial differential equation (PDE), in which the physical stock drift μ is noticeably absent.

However, modern financial economics frequently formulates derivative prices not as solutions to boundary-value partial differential equations, but as discounted expected future payoffs [4, 5]. This dual formulation raises a central probabilistic question: **If dynamic replication renders the no-arbitrage derivative price independent of the physical drift μ, under what probability measure is that expected payoff taken, and why does μ disappear?**

Evaluating the expected terminal payoff under the physical measure $\mathbb{P}$ using the stock's physical drift μ yields an expected terminal stock price $\mathbb{E}^{\mathbb{P}}[S_T \mid \mathcal{F}_t] = S_t e^{\mu(T-t)}$. Simply discounting this physical expectation at the risk-free rate r produces $e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}[H \mid \mathcal{F}_t]$, a quantity that retains an explicit dependence on μ. Discounting a physical expectation at r does not determine an objective, market-clearing no-arbitrage price without introducing additional risk-preference or equilibrium assumptions. Under $\mathbb{P}$, the stock appreciates at rate μ, which reflects an expected return premium $\mu - r$ above the risk-free rate. Discounting that expected growth at r fails to account for the risk premium inherent in the physical trajectory. Conversely, attempting to discount physical expectations using a subjective risk-adjusted rate R introduces individual utility functions into the valuation formula, diverging from the objective, preference-free price established by dynamic replication.

The resolution of this apparent paradox lies in transitioning from the physical measure $\mathbb{P}$ to an **equivalent martingale measure** $\mathbb{Q}$ [4, 1]. Changing measure reweights the probabilities assigned to sample paths without altering which paths are possible, shifting the asset's drift so that all discounted traded asset prices become martingales. This paper develops the continuous-time theory of measure transformation via Girsanov's Theorem [3], proves that discounted asset prices are $\mathbb{Q}$-martingales, evaluates the Black–Scholes European call formula through direct probabilistic integration, and uses the Martingale Representation Theorem [5, 6] to connect risk-neutral valuation with dynamic replication.

2 Physical Asset Dynamics and the Market Price of Risk

Consider a continuous-time financial market defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ for $t \in [0, T]$ satisfying the usual conditions of right-continuity and completeness, where the filtration $\{\mathcal{F}_t\}$ is generated by a standard one-dimensional Brownian motion W_t defined under the physical measure $\mathbb{P}$ [8]. Here, measure-theoretic completeness of the probability space means that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets; this is a standard probabilistic regularity condition and must be kept conceptually distinct from financial market completeness, which concerns the attainability of contingent claims and will be addressed in Section 6.

The market contains two primary traded assets:

1. A risk-free money market account B_t, evolving according to the deterministic differential equation:

$$dB_t = rB_t\,dt, \quad B_0 = 1 \implies B_t = e^{rt}, \tag{1}$$

where $r \geq 0$ is the constant risk-free short rate.

2. A risky stock S_t, whose price dynamics under the physical measure $\mathbb{P}$ follow a geometric Brownian motion [7, 11]:

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t, \quad S_0 > 0, \tag{2}$$

where the physical drift $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$ are constants.

Under the physical measure $\mathbb{P}$, the expected instantaneous rate of return on the stock is μ. When $\mu \neq r$, holding the stock exposes an investor to equity diffusion risk associated with an expected return premium $\mu - r$ above the risk-free rate. The excess return per unit of volatility risk is defined as the **market price of risk** θ:

$$\theta = \frac{\mu - r}{\sigma}. \tag{3}$$

The parameter θ quantifies the physical trade-off between risk and return inherent in the stock process S_t under $\mathbb{P}$ [1].

3 Change of Measure and Girsanov's Theorem

To construct a probability measure under which asset pricing can be conducted consistently without explicit dependence on μ, we perform an absolutely continuous change of measure on $(\Omega, \mathcal{F}_T)$. This construction produces a new probability law $\mathbb{Q}$ under which discounted asset prices become martingales.

3.1 The Radon–Nikodym Density Process

We define the stochastic process Z_t for $0 \le t \le T$ by the exponential expression:

$$Z_t = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right). \tag{4}$$

Applying Itô's Lemma to Z_t yields its stochastic differential representation [7, 8]:

$$dZ_t = -\theta Z_t \, dW_t, \quad Z_0 = 1. \tag{5}$$

Because the stochastic differential in Equation (5) has zero drift under $\mathbb{P}$, Z_t is a continuous local martingale under $\mathbb{P}$. To verify that Z_t is a true martingale, we check Novikov's condition [8, 11]:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \theta^2 \, ds\right)\right] = \exp\left(\frac{1}{2}\theta^2 T\right) < \infty. \tag{6}$$

Since $\theta = (\mu - r)/\sigma$ is constant, the integral in Novikov's condition is simply $\theta^2 T$, so the required expectation equals the finite deterministic quantity $e^{\frac{1}{2}\theta^2 T}$ for every finite T. Consequently, $\mathbb{E}^{\mathbb{P}}[Z_t] = 1$ for all $t \in [0, T]$, and Z_t defines a strictly positive $\mathbb{P}$-martingale. The random variable Z_T serves as the **Radon–Nikodym derivative** for a new probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_T$:

$$Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \quad \text{so that } \mathbb{Q}(A) = \int_A Z_T \, d\mathbb{P} \quad \forall A \in \mathcal{F}_T. \tag{7}$$

3.2 Girsanov's Transformation

The fundamental bridge between probability measures for continuous stochastic processes is provided by Girsanov's Theorem [3].

Theorem 3.1 (Girsanov, 1960). *Let W_t be a standard Brownian motion under $\mathbb{P}$, and let Z_t be the Radon–Nikodym process defined in Equation (4). Then the process $W_t^{\mathbb{Q}}$ defined by*

$$W_t^{\mathbb{Q}} = W_t + \theta t = W_t + \left(\frac{\mu - r}{\sigma}\right) t \tag{8}$$

is a standard Brownian motion under the probability measure $\mathbb{Q}$ on $0 \le t \le T$.

3.3 Transformed Stock Dynamics under $\mathbb{Q}$

Expressing the physical Brownian increments as $dW_t = dW_t^{\mathbb{Q}} - \theta dt$ and substituting into the stock SDE (Equation 2) yields:

$$dS_t = \mu S_t \, dt + \sigma S_t \left(dW_t^{\mathbb{Q}} - \left(\frac{\mu - r}{\sigma}\right) dt\right). \tag{9}$$

Expanding and collecting terms reveals that the physical drift $\mu S_t dt$ is exactly canceled by the drift adjustment $-\sigma S_t \theta dt$:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}. \tag{10}$$

Under $\mathbb{Q}$, the stock price process evolves as a geometric Brownian motion whose mean growth rate is identically equal to the risk-free rate r [11].

3.4 Measure Equivalence and Pathwise Invariance

Girsanov's transformation changes one feature of the model while leaving several others untouched:

1. **What Changes**: The expected rate of return on the stock changes from the physical rate μ to the risk-free rate r. The probability measure shifts probability weights across path space Ω.
2. **What Remains Invariant**:

- **Volatility** σ: The diffusion coefficient σ is completely untouched by the change of measure.
- **Quadratic Variation**: The quadratic variation process $\langle S, S\rangle_t = \int_0^t \sigma^2 S_s^2 \, ds$ is a pathwise property determined by the diffusion parameter σ. It is identical under $\mathbb{P}$ and $\mathbb{Q}$ [8].
- **Null Sets**: Because $Z_T > 0$ $\mathbb{P}$-almost surely, $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent probability measures** ($\mathbb{P} \sim \mathbb{Q}$). This means $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$ for every event $A \in \mathcal{F}_T$.

Equivalence of measures is the core mathematical prerequisite for no-arbitrage pricing [4]. An arbitrage opportunity is defined under the physical measure $\mathbb{P}$ as a self-financing trading strategy with zero initial cost that yields a non-negative payoff with probability 1 and a strictly positive payoff with positive $\mathbb{P}$-probability. Because $\mathbb{P} \sim \mathbb{Q}$, any property that holds almost surely under $\mathbb{Q}$ holds almost surely under $\mathbb{P}$, ensuring that pricing derived under $\mathbb{Q}$ strictly preserves the absence of physical arbitrage.

4 Discounted Asset Prices as $\mathbb{Q}$-Martingales and Risk-Neutral Valuation

The primary motivation for introducing the measure $\mathbb{Q}$ is to turn traded discounted asset price processes into martingales [5, 1].

4.1 The Martingale Property of Discounted Stock Prices

Let $\tilde{S}_t = B_t^{-1} S_t = e^{-rt} S_t$ denote the discounted stock price process. Applying Itô's Product Rule to $\tilde{S}_t = e^{-rt} S_t$ under $\mathbb{Q}$ yields:

$$d\tilde{S}_t = -re^{-rt} S_t \, dt + e^{-rt} dS_t = -re^{-rt} S_t \, dt + e^{-rt} \left(rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}} \right). \tag{11}$$

The deterministic drift terms $-re^{-rt} S_t dt$ and $re^{-rt} S_t dt$ cancel completely, leaving:

$$d\tilde{S}_t = \sigma \tilde{S}_t \, dW_t^{\mathbb{Q}}. \tag{12}$$

In integral form, Equation (12) becomes:

$$\tilde{S}_t = S_0 + \int_0^t \sigma \tilde{S}_s \, dW_s^{\mathbb{Q}}. \tag{13}$$

Explicitly solving Equation (12) gives $\tilde{S}_t = S_0 \exp\left(\sigma W_t^{\mathbb{Q}} - \frac{1}{2}\sigma^2 t\right)$. Because $\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t^2] = S_0^2 e^{\sigma^2 t} < \infty$ for all $t \in [0, T]$, the stochastic integral in Equation (13) is a true square-integrable martingale under $\mathbb{Q}$, not merely a local martingale [8]. Consequently, for any $0 \le t \le s \le T$:

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-rs} S_s \middle| \mathcal{F}_t \right] = e^{-rt} S_t \implies S_t = e^{-r(s-t)} \mathbb{E}^{\mathbb{Q}} \left[S_s \middle| \mathcal{F}_t \right]. \tag{14}$$

Because discounted traded asset prices are martingales under $\mathbb{Q}$, $\mathbb{Q}$ is formally designated as an **Equivalent Martingale Measure (EMM)**.

4.2 General Risk-Neutral Valuation Formula

Consider a European contingent claim maturing at time T with payoff $H = h(S_T)$, where H is an $\mathcal{F}_T$-measurable, square-integrable random variable under $\mathbb{Q}$ ($H \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$). We define the risk-neutral price V_t of the claim at time t by the discounted conditional expectation under $\mathbb{Q}$:

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H\middle|\mathcal{F}_t\right]. \tag{15}$$

Multiplying Equation (15) by e^{-rt} shows that $\tilde{V}_t = e^{-rt}V_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT}H \mid \mathcal{F}_t]$. By the tower property of conditional expectation, $\tilde{V}_t$ is automatically a $\mathbb{Q}$-martingale. Note that establishing V_t as a valid candidate price under an EMM $\mathbb{Q}$ precedes the proof that $\mathbb{Q}$ is unique, which relies on market completeness in Section 6.

5 Probabilistic Evaluation of the European Call Option

We now specialize Equation (15) to a European call option with strike price K and maturity T, whose payoff is $H = (S_T - K)^+$. We show that evaluating this expectation directly under $\mathbb{Q}$ recovers the classical Black–Scholes call pricing formula [2, 11].

5.1 Explicit Distribution of S_T under $\mathbb{Q}$

Integrating SDE (10) under $\mathbb{Q}$ provides the explicit solution for S_T given $\mathcal{F}_t$:

$$S_T = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\,Y\right), \tag{16}$$

where $Y = \frac{W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}}{\sqrt{T-t}}$ is a standard normal random variable under $\mathbb{Q}$ ($Y \sim \mathcal{N}(0,1)$).

Substituting Equation (16) into valuation Equation (15), the option value $C_t = C(S_t, t)$ is written as an integral against the standard normal density $\phi(y) = (2\pi)^{-1/2}e^{-y^2/2}$:

$$C_t = e^{-r(T-t)}\int_{-\infty}^{\infty}\left(S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\,y\right) - K\right)^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy. \tag{17}$$

5.2 Integration Domain and Splitting

The payoff term inside the integral is strictly positive if and only if $S_T > K$. Solving for y:

$$S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\,y\right) > K \iff y > -d_2, \tag{18}$$

where d_2 is defined by:

$$d_2 = \frac{\ln(S_t/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{19}$$

Restricting the domain of integration to $y \in (-d_2, \infty)$, integral (17) splits naturally into two terms, I_1 and I_2:

$$C_t = I_1 - I_2, \tag{20}$$

where:

$$I_1 = S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2 + \sigma\sqrt{T-t}\,y - \frac{1}{2}\sigma^2(T-t)\right) dy, \tag{21}$$

$$I_2 = Ke^{-r(T-t)} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy. \tag{22}$$

5.3 Evaluating I_2 and I_1

For I_2, standard Gaussian symmetry gives $\int_{-d_2}^{\infty}(2\pi)^{-1/2}e^{-y^2/2}dy = N(d_2)$, where $N(x)$ denotes the standard normal cumulative distribution function $N(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}e^{-z^2/2}dz$:

$$I_2 = Ke^{-r(T-t)}N(d_2). \tag{23}$$

For I_1, completing the square in the exponent of the integrand yields:

$$-\frac{1}{2}y^2 + \sigma\sqrt{T-t}\,y - \frac{1}{2}\sigma^2(T-t) = -\frac{1}{2}\left(y - \sigma\sqrt{T-t}\right)^2. \tag{24}$$

Making the substitution $z = y - \sigma\sqrt{T-t}$ ($dz = dy$), the lower integration limit becomes $-d_2 - \sigma\sqrt{T-t} = -d_1$, where d_1 is defined as:

$$d_1 = d_2 + \sigma\sqrt{T-t} = \frac{\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{25}$$

Evaluating I_1 under this change of variable gives:

$$I_1 = S_t \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = S_t N(d_1). \tag{26}$$

Subtracting I_2 from I_1 recovers the exact Black–Scholes European call pricing formula:

$$C(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2). \tag{27}$$

5.4 Why μ Disappears and the Precise Meaning of "Risk-Neutral"

This direct expectation calculation reveals why the physical drift μ disappears from derivative pricing:

1. **Why μ Disappears**: Under the physical measure $\mathbb{P}$, the stock has an expected excess return $\mu - r = \theta\sigma$ in the model. However, option pricing does not evaluate unhedged equity risk. Because an option can be perfectly hedged by dynamic trading in the stock and money market account, the option's cost is governed entirely by the cost of financing that replicating strategy at the risk-free interest rate r. Probabilistically, Girsanov's Theorem adjusts path probabilities by subtracting the market price of risk θ from the Brownian motion W_t, replacing μ with r.

2. **Precise Meaning of "Risk-Neutral"**: The term "risk-neutral measure" defines a **formal mathematical property of the pricing measure** $\mathbb{Q}$—specifically, that under $\mathbb{Q}$, all traded discounted asset price processes are martingales with expected growth rate r. It does **not** imply that real-world investors are risk-neutral or indifferent to risk. Market participants may be risk-averse under $\mathbb{P}$, but no-arbitrage replication forces derivative valuation to coincide mathematically with conditional expectation under $\mathbb{Q}$.

6 Market Completeness and the Martingale Representation Bridge

We now connect probabilistic valuation under $\mathbb{Q}$ to the continuous-time dynamic replication developed in the previous paper, and establish the uniqueness of $\mathbb{Q}$.

6.1 Financial Market Completeness and Uniqueness of $\mathbb{Q}$

In financial mathematics, a financial market is defined as **complete** if every square-integrable contingent claim $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$ is attainable—meaning there exists a self-financing trading strategy that perfectly replicates H at maturity T [5, 6].

Under standard admissibility and integrability conditions, the fundamental connection between arbitrage, measure uniqueness, and market completeness is codified by the Fundamental Theorems of Asset Pricing (Harrison, Kreps, and Pliska) [4, 5, 6]:

- **First Fundamental Theorem of Asset Pricing**: Under standard admissibility conditions, a continuous-time market model is free of arbitrage if and only if there exists at least one equivalent martingale measure $\mathbb{Q}$.
- **Second Fundamental Theorem of Asset Pricing**: An arbitrage-free continuous-time market model is **complete** if and only if the equivalent martingale measure $\mathbb{Q}$ is **unique**.

In the Black–Scholes model, uncertainty is generated by a one-dimensional Brownian filtration $\{\mathcal{F}_t\}$, and trading is restricted to one non-degenerate risky stock (S_t, $\sigma > 0$) and a risk-free money market account (B_t). By the Brownian Martingale Representation Theorem [8], every square-integrable $\mathbb{Q}$-martingale can be represented as an Itô integral with respect to $W_t^{\mathbb{Q}}$. Because $\sigma > 0$, the stock price diffusion $\sigma S_t dW_t^{\mathbb{Q}}$ spans the single Brownian source of risk, establishing that every square-integrable claim H is attainable. Thus, the Black–Scholes market is complete. By the Second Fundamental Theorem of Asset Pricing, this financial completeness implies that the equivalent martingale measure $\mathbb{Q}$ is **unique**. The existence of a unique algebraic solution $\theta = (\mu - r)/\sigma$ reflects this structural completeness rather than serving as its sole proof.

6.2 The Martingale Representation Theorem and Replicating Strategies

The mathematical bridge connecting the risk-neutral expectation V_t to the dynamic replication portfolio Δ_t is the **Martingale Representation Theorem** [5, 8].

Let $M_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT}H \mid \mathcal{F}_t] = e^{-rt}V_t$ denote the discounted option value process under $\mathbb{Q}$. Because M_t is a square-integrable martingale with respect to the filtration generated by the Brownian motion $W_t^{\mathbb{Q}}$ under $\mathbb{Q}$, the Martingale Representation Theorem guarantees the existence of a unique, adapted process ψ_t such that M_t can be represented as an Itô integral with respect to $W_t^{\mathbb{Q}}$:

$$M_t = M_0 + \int_0^t \psi_s \, dW_s^{\mathbb{Q}}. \tag{28}$$

To convert this Brownian representation into a tradeable asset representation, we recall from Equation (12) that $d\tilde{S}_t = \sigma \tilde{S}_t \, dW_t^{\mathbb{Q}}$, which implies $dW_t^{\mathbb{Q}} = \frac{d\tilde{S}_t}{\sigma \tilde{S}_t}$. Substituting this relation into Equation (28) yields:

$$M_t = M_0 + \int_0^t \phi_s \, d\tilde{S}_s, \quad \text{where } \phi_s = \frac{\psi_s}{\sigma \tilde{S}_s}. \tag{29}$$

Converting Equation (29) back to undiscounted value $V_t = e^{rt} M_t$ using Itô's Product Rule gives:

$$dV_t = rV_t\,dt + \phi_t\,(dS_t - rS_t\,dt) = \phi_t\,dS_t + (V_t - \phi_t S_t) r\,dt. \tag{30}$$

Equation (30) is precisely the self-financing portfolio condition for a replicating strategy holding $\Delta_t = \phi_t$ shares of stock and allocating $V_t - \phi_t S_t$ to the money market account.

By Markov properties and Itô's Lemma applied to $V_t = C(S_t, t)$, the integrand ϕ_t is identified explicitly as the option's delta:

$$\Delta_t = \phi_t = \frac{\partial C(S_t, t)}{\partial S_t} = N(d_1). \tag{31}$$

This establishes the complete equivalence bridge:

$$\begin{array}{ccccc} \text{Claim } H & \longrightarrow & \text{Discounted Martingale } M_t & \longrightarrow & \text{Integral Representation } \int \phi\, d\tilde{S} \\ & & \big\downarrow & & \big\downarrow \\ & & \text{Risk-Neutral Price } V_t & \longleftrightarrow & \text{Replicating Strategy } \Delta_t = \frac{\partial C}{\partial S} \end{array}$$

7 Conclusion

The Black–Scholes framework admits two mathematically equivalent views of no-arbitrage pricing. The replication approach eliminates the stock's diffusion risk and determines the option price without reference to the physical drift μ [10]. The probabilistic approach reaches the same price by changing from the physical measure $\mathbb{P}$ to the equivalent martingale measure $\mathbb{Q}$, under which discounted traded asset prices are martingales.

Girsanov's Theorem explains precisely how the measure change replaces the drift μ with the risk-free rate r while preserving volatility, path continuity, and null sets. Direct evaluation under $\mathbb{Q}$ recovers the Black–Scholes European call formula, while the Martingale Representation Theorem connects the resulting risk-neutral price to its replicating strategy. In the complete Black–Scholes market, risk-neutral valuation and dynamic replication therefore represent the same unique no-arbitrage price from two mathematical perspectives.

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